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ABSTRACT

A classic and important finding in the empirical analysis of employment and output is that the short run elasticity of demand with respect to output is smaller than unity and is also less than the long run elasticity: this phenomenon has been called "short run increasing returns to labor" (SRIRL).

In this paper we show, both analytically and empirically, that SRIRL can be viewed as the natural outcome of a dynamic cost minimization process. We analyze SRIRL using a dynamic model of factor demands which explicitly generates short- and long-run demand equations for variable and quasi-fixed inputs, where the quasi-fixed inputs are subject to increasing marginal internal costs of adjustment. Speeds of adjustment of quasi-fixed inputs to their long run equilibrium levels are endogenous, rather than constant parameters. We devote particular attention to the notion of labor hoarding, and demonstrate analytically that labor hoarding is neither necessary nor sufficient for SRIRL. These analytical results are illustrated with empirical findings based on annual U.S. manufacturing data, 1952-71.

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I. Introduction

Since the pioneering work of Thor Hultgren [1960], Edwin Kuh [1960, 1965], Arthur Okun [1962] and Frank Brechling [1965], a great deal of econometric research has focussed attention on the short run relationship between output and employment. The most salient finding in this empirical literature has been called "short run increasing returns to labor" (SRIRL): the short run elasticity of demand for aggregate labor with respect to output is smaller than unity, and is also less than the long run elasticity. This implies that the short run average productivity of aggregate labor is procyclical.[1]

The SRIRL finding is quite robust empirically: it occurs both for aggregate employment and, to a slightly less extent, for aggregate manhours (Kuh [1965], Brechling [1965]), is pervasive in two-digit U.S. manufacturing industries (Kuh [1965]), cannot be explained adequately by output compositional changes among industries during the business cycle (Kuh [1965], William Nordhaus [1972], Knut Mork [1977]), and occurs in virtually all OECD countries (Brechling and Peter O'Brien [1965]).

Some have viewed the SRIRL finding as being inconsistent with the traditional neoclassical theory of the firm. Consider, for example, a firm initially in equilibrium producing output with a variable input labor and a fixed (in the short run) factor capital. If the short run production function were of the normal shape with diminishing returns to the variable factor labor, then as output increased the short run elasticity of demand for labor with respect to output would be greater than unity, and average cost would rise with output. However, the empirical finding of SRIRL directly contradicts this simple theory.

Attempts to resolve this apparent paradox have taken several different courses. One approach, that of labor hoarding, is based on empirical evidence

that the SRIRL finding was much stronger for nonproduction than for production workers (Otto Eckstein and Thomas Wilson [1964] and Kuh [1965]), and on the related notion that certain tapes of skilled or overhead labor can be viewed as quasi-fixed inputs (Walter Oi [1962]). In a rigorous development of the labor hoarding argument, Robert M. Solow [1968] specified that substantial increasing marginal adjustment costs occur with rapid changes in the level of employment or average weekly hours, particularly for overhead workers. As a consequence, the dynamic cost minimization solution for the firm facing a known cyclical demand for its output will be to hoard labor in the downturn of the business cycle. This yields a procyclical pattern of output per manhour, even when the firm's underlying production function is well-behaved. To the best of our knowledge, however, no empirical research has appeared based on a Solow-type model of explicit dynamic cost minimization.[2]

An alternative approach to unravelling the SRIRL paradox involves placing relatively less attention on the deterministic economic structure and greater emphasis on stochastic considerations. In this alternative framework, the firm is envisaged as minimizing expected discounted costs where, for example, expected output contains a known trend term and a serially correlated stochastic component. Christopher A. Sims [1974] has developed and estimated such a time series model where demand for production labor is affected by output, but not by relative factor prices or size of the capital stock. Sims obtains empirical support for Granger causality running in the direction of exogenous output to endogenous employment, but finds constant rather than increasing short run returns to production labor; manhours of production workers follow an almost proportional cumulative response to output, with the strongest response occurring in the contemporaneous month and the full response being

completed within six to nine months. It should be noted that while Sims finds very little support for SRIRL occurring with production workers, nonproduction ("white collar") workers are excluded from his sample.

A somewhat more complex labor demand model has been formulated and estimated by Thomas J. Sargent [1978], where stochastic processes for real wages and the average productivity relations follow a stationary Markov scheme. In Sargent's model, employment demand is affected by output and real wages, but the capital stock is assumed to remain at its 1947I value throughout the 1947I - 1972V time period. Like Sims, Sargent finds evidence supporting the view that output (and real wages) are exogenous, while employment is endogenous.

Currently research is underway to enrich the economic structural content of interrelated factor demand models, while simultaneously allowing for multiple variable stochastic rational expectations; see, for example, Lars P. Hansen and Thomas J. Sargent [1979, 1980]. Difficulties in obtaining convenient mathematical representations, even for linear stochastic models, have to this point precluded empirical implementation.

In this paper we draw from both approaches, but focus primarily on the deterministic economic structure of SRIRL rather than on the stochastic expectations process. In particular, following Solow [1968], we examine the role of labor hoarding in SRIRL by applying a dynamic model of explicit cost minimization with internal costs of adjustment to quasi-fixed factors, which treats short and long run within a unified Marshallian framework. Our empirically implementable multiple input dynamic model extends earlier work by Robert E. Lucas [1967a,b], Arthur B. Treadway [1971, 1974], L.J. Lau [1976] and E.R. Berndt, M.A. Fuss and Leonard Waverman [1977, 1980], and makes use of the Sims-Sargent result that Granger causality runs from output and relative factor prices to employment. A surprising analytical result we obtain is

that labor hoarding is neither necessary nor sufficient for SRIRL to occur. This analytical finding is illustrated empirically for U.S. manufacturing, 1952-71.

In our view, future research should attempt to synthesize the economic content of our deterministic dynamic cost minimization model with the approach that attributes some degree of rationality to the stochastic expectations formation process.

II. Theoretical Model and Econometric Specification

Assume there are I quasi-fixed inputs x_i $(i = 1,\cdots,I)$ and J variable inputs v_j $(j = 1,\cdots,J)$ entering the firm's production process. Following Treadway [1974] and E.R. Berndt, M.A. Fuss and Leonard Waverman [1977, 1980], we specify that the quasi-fixed inputs are subject to increasing internal costs of adjustment, i.e., as purchases of each quasi-fixed input increase, the amount of foregone output rises, implying that the fixed factor is available at increasing unit price.[3] In contrast, the variable inputs are available at constant unit price. Specifically, let the vector of exogenous prices of variable inputs be $\hat{w} = \{\hat{w}_j\}$, and let the variation of the quasi-fixed inputs x_i be possible at a cost $c_i(\dot{x}_i)$, where

$$c_i(0) = 0 \;, \qquad c_i'(\dot{x}_i) > 0 \;, \qquad c_i''(\dot{x}_i) > 0 \;. \qquad \ldots .(1)$$

The optimization problem facing the firm is to choose $x_i(t)$, the vector of quasi-fixed factors, and $v_j(t)$, the vector of variable inputs, to minimize the present value of the cost of producing a given flow of output, $Q(t)$, subject to a production function constraint $Q(t) = F(v,x,\dot{x})$.

The present value of cost at time $t = 0$ is given by

$$V(0) = \int_0^\infty e^{-rt} \left(\sum_j \hat{w}_j v_j + \sum_i \hat{q}_i z_i \right) dt \qquad \ldots . (2)$$

where r is the firm's discount rate, $\hat{q}_i$ is the asset price of new quasi-fixed inputs, and $z_i = \dot{x}_i + d_i x_i$, where d_i is the depreciation rate on the i^{th} quasi-fixed input stock and z_i therefore represents the gross addition to the stock of x_i. We assume that factor prices are expected to increase at a constant exponential rate, implying that relative factor price expectations are static, and that the constant rate of inflation can be factored out to transform r into a real discount rate. We also assume static expectations with respect to r and Q. The minimization problem is solved by choosing the time paths v(t), x(t) so as to minimize V(0), given the initial stocks of the quasi-fixed inputs x(0), and v(t), x(t) > 0.

The factor requirements function for $v_j(t)$ is obtained by inverting the production function constraint:

$$v_1(t) = F^{-1}[v_2(t), \cdots, v_J(t), x(t), \dot{x}(t), Q(t)] . \qquad \ldots . (3)$$

Substituting (3) into (2) and solving the optimization problem for the cost minimizing variable inputs $\bar{v}_j(t)$ conditional on x(t), $\dot{x}(t)$ and Q(t), we obtain the normalized restricted cost function

$$G[w(t), x(t), \dot{x}(t), Q(t)] = \sum_j w_j \bar{v}_j \qquad \ldots . (4)$$

where w_j is the normalized factor price $w_j = \hat{w}_j / \hat{w}_1$. L.J. Lau [1976] has shown that under reasonable assumptions G satisfies the following regularity conditions:

(i) G is increasing in w and decreasing in x

(ii) G is concave in w

(iii) G is convex in x and monotonically increasing in Q

(iv) $\frac{\partial G}{\partial w_j} = \bar{v}_j$, the conditional (short run) cost minimizing input level for the j^{th} variable input

(v) $-\frac{\partial G}{\partial x_i} = u_i$, the normalized shadow price of the service flow of the quasi-fixed input x_i, which in the simplest case equals $q_i(r+d_i)$. . . . . (5)

Note that G is a variable cost function assuming short run optimizing behavior conditional on Q(t), x(t) and $\dot{x}(t)$. Hence, the second step in the optimization process is to find among all the possible $G[w(t),x(t),\dot{x}(t),Q(t)]$ combinations, that x(t), $\dot{x}(t)$ minimizing the present value of costs,

$$V(0) = \int_0^\infty e^{-rt}\left(G[w,x,\dot{x},Q] + \sum_i q_i z_i\right) dt$$

$$= \int_0^\infty e^{-rt}\left(G[w,x,\dot{x},Q] + \sum_i u_i x_i\right) dt - \sum_i q_i x_i(0) \quad . \qquad \ldots\ldots(6)$$

Minimizing (6) with respect to x(t) is equivalent to minimizing (2) with respect to x(t) and the variable inputs v(t), since (6) incorporates the optimal variable factors conditional on x(t).

Minimization of (6) yields the Euler first order conditions

$$-G_x - rG_{\dot{x}} - u + G_{\dot{x}\dot{x}}\ddot{x} + G_{x\dot{x}}\dot{x} = 0 \qquad \ldots\ldots(7)$$

where the x, $\dot{x}$ subscripts denote derivatives and $\ddot{x}$ is the second partial derivative with respect to time. The steady state solution satisfies

$$-G_x(w,x^*) - rG_{\dot{x}}(w,x^*) - u = 0 \quad , \qquad \ldots .(8)$$

x^* being unique as long as $\left| -G^*_{xx} - rG^*_{x\dot{x}} \right| \neq 0$, where $*$ indicates evaluation at $x = x^*$ and $\dot{x} = 0$.

Treadway has shown that this model can be interpreted as a flexible accelerator type model, since the demand for x_i can be determined from (7) or (8) as an approximate solution to the differential equation system

$$\dot{x} = M^*(x^* - x) \quad , \qquad \ldots .(9)$$

where M^* is determined from the solution to the quadratic form

$$-G^*_{\dot{x}\dot{x}} M^{*2} - rG^*_{x\dot{x}} M^* + (G^*_{xx} + rG^*_{x\dot{x}}) = 0 \quad . \qquad \ldots .(10)$$

In the special case of only one quasi-fixed input Treadway obtained

$$\dot{x}_1 = m^*_{11} (x^*_1 - x_1) \qquad \ldots .(11)$$

where at the stationary point when $G^*_{x\dot{x}} = 0$,

$$m^*_{11} = -\tfrac{1}{2} [r - (r^2 + 4G_{x_1x_1}/G_{\dot{x}_1\dot{x}_1})^{\frac{1}{2}}] \quad . \qquad \ldots .(12)$$

For empirical implementation, we can specify a functional form for the production technology as embodied in the normalized variable cost function

$$G(w,x,\dot{x},Q,t) = \sum_j \hat{w}_j \bar{v}_j = \bar{v}_1 + \sum_{j=2}^{J} w_j \bar{v}_j \quad .$$

Short run or conditional input demand equations for v_j $(j = 2,\cdots,J)$ can be obtained by differentiation as in (5-iv). The short run demand for v_1 can

then be computed as $G(\cdot) - \sum_{j=2}^{J} w_j \bar{v}_j$, and the $\dot{x}$ equations can be derived from (8) and (9) above to yield a system of J + I estimable cost-minimizing demand equations.

From these equations, expressions for short, intermediate and long run elasticities can be derived which completely summarize the dynamic behavior of factor demands. Following the Marshallian framework, short run elasticities are defined as those obtained when x is fixed, intermediate run as the impact when x has adjusted partially as determined by M*, and long run as the response when x has adjusted fully to x* . Thus in the case of one quasi-fixed input the short run price elasticities for variable inputs can be calculated as

$$\varepsilon^S_{v_j w_k} \equiv \left(\frac{w_k}{v_j}\right)\left(\left.\frac{\partial v_j}{\partial w_k}\right|_{x_1 = \bar{x}_1}\right) , \qquad \ldots .(13)$$

the corresponding intermediate (one-period) run elasticities as

$$\varepsilon^I_{v_j w_k} = \left(\frac{w_k}{v_j}\right) \cdot \left(\left.\frac{\partial v_j}{\partial w_k}\right|_{x_1 = \bar{x}_1} + m^*_{11} \cdot \frac{\partial v_j}{\partial x^*_1}\frac{\partial x^*_1}{\partial w_k}\right) \qquad \ldots .(14)$$

and long run elasticities (using the sum of a geometric progression rule) as

$$\varepsilon^L_{v_j w_k} = \left(\frac{w_k}{v_j}\right) \cdot \left(\left.\frac{\partial v_j}{\partial w_k}\right|_{x_1 = \bar{x}_1} + \frac{\partial v_j}{\partial x^*_1} \cdot \frac{\partial x^*_1}{\partial w_k}\right), \quad j,k = 1,\cdots,J . \qquad \ldots .(15)$$

The short run elasticities of the quasi-fixed factors are of course equal to zero, but intermediate and long run elasticities are non-zero.

Because of our particular interest in the phenomenon of short run increasing returns to labor, the derivation of output elasticities is

especially important. The short and long run output elasticities for v_j and x_1 are expressed as follows:

$$\varepsilon^S_{v_jQ} = (Q/v_j) \cdot (\partial v_j/\partial Q \Big|_{x_1 = \bar{x}_1})$$

$$\varepsilon^S_{x_1Q} = (Q/x_1) \cdot (\partial x_1/\partial Q \Big|_{x_1 = \bar{x}_1}) = 0 \qquad \ldots .(16)$$

$$\varepsilon^L_{v_jQ} = (Q/v_j) \left(\frac{\partial v_j}{\partial Q} \Big|_{x_1 = \bar{x}_1} + \frac{\partial v_j}{\partial x_1^*} \cdot \frac{\partial x_1^*}{\partial Q} \right)$$

$$\varepsilon^L_{x_1Q} = (Q/x_1^*) \left(\frac{\partial x_1^*}{\partial Q} \right) . \qquad \ldots .(17)$$

Other elasticities can be calculated in an analogous manner.

The empirical finding of SRIRL can easily be interpreted in terms of this model. A particularly useful way of expressing the SRIRL phenomenon, (letting $v_1 \equiv L \equiv$ labor), is that $\varepsilon^S_{LQ} < 1$ and $\varepsilon^S_{LQ} < \varepsilon^L_{LQ}$, i.e., the short run elasticity of demand for labor with respect to output is less than unity and smaller than the long run elasticity. Denoting the single quasi-fixed input as K for physical capital, we can use (16) and (17) and write the difference between the long and short run labor demand elasticities as

$$\varepsilon^L_{LQ} - \varepsilon^S_{LQ} = (Q / L) \cdot \left(\frac{\partial L}{\partial K^*} \cdot \frac{\partial K^*}{\partial Q} \right) \qquad \ldots .(18)$$

where K* is the long run equilibrium level of capital. Since Q, L, and $\partial K^*/\partial Q$ are positive, the sign of (18) depends on the sign of $\partial L / \partial K^*$. It follows, therefore, that

$$\varepsilon^L_{LQ} \gtrless \varepsilon^S_{LQ} \quad \text{according as} \quad \frac{\partial L}{\partial K^*} \gtrless 0 . \qquad \ldots .(19)$$

In order for the long run labor demand elasticity to be larger than the short run elasticity, it is necessary and sufficient in this case that $\frac{\partial L}{\partial K^*} > 0$, i.e., that L and K* be Hicks-Allen complementary inputs. If there were only two inputs, $\varepsilon_{LQ}^{L} \leq \varepsilon_{LQ}^{S}$, since concavity of the production function rules out complementarity.

The fact that complementarity between K and L will generate $\varepsilon_{LQ}^{L} > \varepsilon_{LQ}^{S}$ is quite intuitive. In response to a short run increase in demand for output, the firm will attempt to increase demands for the variable inputs, since K is fixed in the short run, i.e., the firm will attempt to substitute variable inputs for the fixed factor. But this increase in the demand for variable inputs will not necessarily be equi-proportional across all inputs. The variable inputs that are most substitutable with the quasi-fixed capital input will be utilized the most intensively in the short run, reducing their average and marginal productivity, but increasing the average and marginal productivity of the fixed factor and its complements.[4] As time passes, and if output remains at its increased level, the firm will adjust capital to a higher level, reduce the intensity of those inputs highly substitutable with capital, and increase the intensity (reduce the average product) of those inputs complementary with capital.

In summary, in the above model with a single quasi-fixed input, $\varepsilon_{LQ}^{S} < \varepsilon_{LQ}^{L}$ does not require labor hoarding, but instead relies on a complementary relationship between aggregate labor and the single quasi-fixed capital input (K). *A priori*, that aggregate labor and capital are complementary inputs is rather implausible. However, if aggregate labor were disaggregated into production ("unskilled," hereafter U) and non-production ("skilled," (S)) labor, and if K and S were complementary inputs while K and U were substitutable, then $\varepsilon_{SQ}^{L} > \varepsilon_{SQ}^{S}$,

$\varepsilon^L_{UQ} < \varepsilon^S_{UQ}$. Moreover, if $\varepsilon^L_{SQ} > \varepsilon^S_{SQ}$ (as a consequence of capital-skill complementarity) dominated $\varepsilon^L_{UQ} < \varepsilon^S_{UQ}$ (the result of capital-unskilled labor substitutability), it would be possible that for measured aggregate labor $\varepsilon^L_{LQ} > \varepsilon^S_{LQ}$. Hence, evidence consistent with SRIRL could result from an aggregation phenomenon due to capital-skilled labor complementarity and capital-unskilled labor substitutability.

In order to illustrate the above model empirically, a functional form must be specified for G and the continuity assumptions incorporated in the theoretical model must be modified to conform with data constraints. We assume that (i) continuous changes in quasi-fixed inputs $\dot{x}_i$ can be represented by discrete changes Δx_i, and that (ii) output in period t is produced by quasi-fixed inputs in place at the beginning of the period. (iii) Regarding functional form, we have modified the quadratic normalized restricted cost function of Berndt, Fuss and Waverman [1977, 1980] so that coefficients of the demand equations represent effects of price changes on input-output ratios rather than on input levels. This functional form for the normalized variable cost function with I quasi-fixed inputs, J variable inputs, and with long run constant returns to scale imposed is:

$$G = \bar{v}_1 + \sum_{j=2}^{J} w_j \bar{v}_j = Q\Big(\alpha_0 + \alpha_{0t} \cdot t + \sum_{j=2}^{J} \alpha_j \cdot w_j$$

$$+ \frac{1}{2} \sum_{j=2}^{J} \sum_{\ell=2}^{J} \gamma_{j\ell} \cdot w_j \cdot w_\ell + \sum_{j=2}^{J} \alpha_{jt} \cdot w_j \cdot t\Big)$$

$$+ \sum_{i=1}^{I} \alpha_i \cdot x_i + \sum_{i=1}^{I} \dot{\alpha}_i \Delta x_i + \frac{1}{2} \sum_{i=1}^{I} \sum_{k=1}^{I} \gamma_{ik} \frac{x_i \cdot x_k}{Q}$$

$$+ \sum_{i=1}^{I} \sum_{j=2}^{J} \gamma_{ij} \cdot w_j \cdot x_i + \sum_{i=1}^{I} \alpha_{it} \cdot x_i \cdot t$$

$$+ \frac{1}{2} \sum_{i=1}^{I} \sum_{k=1}^{I} \ddot{\gamma}_{ik} \cdot \frac{\Delta x_i \cdot \Delta x_k}{Q} + \sum_{i=1}^{I} \sum_{j=2}^{J} \dot{\gamma}_{ij} \cdot w_j \cdot \Delta x_i$$

$$+ \sum_{i=1}^{I} \dot{\alpha}_{it} \cdot \Delta x_i \cdot t + \sum_{i=1}^{I} \sum_{k=1}^{I} \dot{\gamma}_{ik} \cdot \Delta x_i \cdot x_k \qquad \ldots .(20)$$

where Q is gross output, x_i and v_j are the quasi-fixed and variable inputs, respectively, t represents the state of technology as measured by time, i.e., $t = 1,\cdots,T$ and where $\gamma_{j\ell} = \gamma_{\ell j}$, $\gamma_{ik} = \gamma_{ki}$, $\ddot{\gamma}_{ik} = \ddot{\gamma}_{ki}$, and $\dot{\gamma}_{ik} = \dot{\gamma}_{ki}$.

That portion of (20) representing internal costs of adjustment for the i^{th} quasi-fixed input is

$$c(\Delta x_i) = \dot{\alpha}_i \cdot \Delta x_i + \sum_{\substack{k \neq i \\ k=1}}^{I} \ddot{\gamma}_{ik} \cdot \frac{\Delta x_i \cdot \Delta x_k}{Q} + \frac{1}{2} \ddot{\gamma}_{ii} \frac{(\Delta x_i)^2}{Q}$$

$$+ \sum_{j=2}^{J} \dot{\gamma}_{ij} \cdot w_j \cdot \Delta x_i + \dot{\alpha}_{it} \cdot \Delta x_i \cdot t$$

$$+ \sum_{k=1}^{I} \dot{\gamma}_{ik} \cdot \Delta x_i \cdot x_k \, , \qquad i = 1,\cdots,I \qquad \ldots .(21)$$

At a stationary point where Δx_i must equal zero for all $i = 1,\cdots,I$, marginal adjustment costs will be zero when

$$c'(\Delta x_i) = \dot{\alpha}_i + \sum_{k=1}^{I} \ddot{\gamma}_{ik} \cdot \frac{\Delta x_k}{Q} + \sum_{j=2}^{J} \dot{\gamma}_{ij} \cdot w_j + \dot{\alpha}_{it} \cdot t$$

$$+ \sum_{k=1}^{I} \dot{\gamma}_{ik} \cdot x_k = 0 , \qquad i = 1,\cdots,I. \qquad \ldots.(22)$$

This stationary condition will hold for any w_j, x_k, and t only if the following restrictions are imposed:

$$\dot{\alpha}_i = \dot{\gamma}_{ij} = \dot{\alpha}_{it} = \dot{\gamma}_{ik} = 0 , \qquad i,k = 1,\cdots,I; \; j = 2,\cdots,J . \qquad \ldots.(23)$$

When the restrictions (23) are imposed, the normalized variable cost function (20) collapses to

$$G = \bar{v}_1 + \sum_{j=2}^{J} w_j \cdot \bar{v}_j = Q\Big(\alpha_0 + \alpha_{0t} \cdot t + \sum_{j=2}^{J} \alpha_j \cdot w_j$$

$$+ \frac{1}{2} \sum_{j=2}^{J} \sum_{\ell=2}^{J} \gamma_{j\ell} \cdot w_j \cdot w_\ell + \sum_{j=2}^{J} \alpha_{jt} \cdot w_j \cdot t\Big) + \sum_{i=1}^{I} \alpha_i \cdot x_i$$

$$+ \frac{1}{2} \sum_{i=1}^{I} \sum_{k=1}^{I} \gamma_{ik} \cdot \frac{x_i \cdot x_k}{Q} + \sum_{i=1}^{I} \sum_{j=2}^{J} \gamma_{ij} \cdot w_j \cdot x_i$$

$$+ \sum_{i=1}^{I} \alpha_{it} \cdot x_i \cdot t + \frac{1}{2} \sum_{i=1}^{I} \sum_{k=1}^{I} \ddot{\gamma}_{ik} \cdot \frac{\Delta x_i \cdot \Delta x_k}{Q} . \qquad \ldots.(24)$$

As noted above in (5-iv), the conditional (short run) cost minimizing demand function for the variable inputs can be derived as $\bar{v}_j = \partial G/\partial w_j$. In the

present context, the J-1 input-output demand functions for variable inputs are (after dividing by Q)

$$\bar{v}_j/Q = \alpha_j + \sum_{\ell=2}^{J} \gamma_{j\ell} \cdot w_\ell + \alpha_{jt} \cdot t + \sum_{i=1}^{I} \gamma_{ij} \cdot (x_i/Q) \qquad j = 2,\cdots,J. \quad . \quad .(25)$$

Furthermore, since $G(\cdot) = \bar{v}_1 + \sum_{j=2}^{J} w_j \bar{v}_j$, [$G(\cdot)$ incorporates conditional cost minimization], $\bar{v}_1$ can be derived as $\bar{v}_1 = G(\cdot) - \sum_{j=2}^{J} w_j \cdot \bar{v}_j$. Dividing $G(\cdot)$ by Q and substituting in from (25) we obtain the short-run conditional input-output demand equation for the first (normalized) input as

$$\bar{v}_1/Q = \alpha_0 + \alpha_{0t} \cdot t - \frac{1}{2} \sum_{j=2}^{J} \sum_{\ell=2}^{J} \gamma_{j\ell} \cdot w_j \cdot w_\ell + \sum_{i=1}^{I} \alpha_i (x_i/Q)$$

$$+ \sum_{i=1}^{I} \alpha_{it} \cdot (x_i/Q) \cdot t + \frac{1}{2} \sum_{i=1}^{I} \sum_{k=1}^{I} \gamma_{ik} \cdot x_i \cdot (x_k/Q)$$

$$+ \frac{1}{2} \sum_{i=1}^{I} \sum_{k=1}^{I} \ddot{\gamma}_{ik} \cdot \Delta x_i \cdot \Delta x_k/Q^2 \quad . \qquad . \quad . \quad . \quad .(26)$$

Together (25) and (26) yield J estimable short run input-output demand equations for the J variable inputs.

The derivation of demand equations for the I quasi-fixed inputs is more complex. Consider for the moment the case of only one quasi-fixed input, x_1 (the case of more than one quasi-fixed input is discussed in Section III below). The long run equilibrium level of $x_1 = x_1^*$ can be derived from the Euler necessary conditions for cost minimization (8) and the normalized variable cost function (24) yielding

$$x_1^* = \left(\frac{-1}{\gamma_{11}}\right) \cdot \left(\alpha_1 + \sum_j \gamma_{1j} \cdot w_j + \alpha_{1t} \cdot t + u_1\right) \cdot Q \qquad \ldots . (27)$$

where u_1 is the rental or service flow price of the quasi-fixed input. Using (27), the expression for m_{11}^* from (12) and (13), and the differential equation (11) which in discrete time is given by

$$\Delta x_{1t} \equiv x_{1,t+1} - x_{1,t} = m_{11}^*(x_{1,t+1}^* - x_{1,t}) \quad , \qquad \ldots . (28)$$

the optimal input-output demand equation for x_1 becomes

$$\frac{\Delta x_{1t}}{Q_t} = -\frac{1}{2}\left[r_t - (r_t^2 + 4\gamma_{11}/\ddot{\gamma}_{11})^{1/2}\right] \cdot \left[(-1/\gamma_{11}) \cdot (\alpha_1 + \sum_j \gamma_{1j} w_{jt} + \alpha_{1t} \cdot t + U_{1t}) - \frac{x_{1,t}}{Q_t}\right] . \qquad \ldots . (29)$$

Thus our entire system of estimable demand equations consists of the J variable input-output demand equations (25) and (26), and the input-output demand equation for the quasi-fixed input (29).

In the next section of this paper we shall report results based on two applications of the above model. In the first KLEM model, the quasi-fixed input is physical capital (K), the first normalized variable input is aggregate labor (L), and the remaining inputs are aggregate energy (E) and aggregate non-energy intermediate materials (M). We denote the normalized prices of these inputs as P_K, P_L, P_E, and P_M, respectively. The four estimating equations are based on (25), (26), and (29).

We append an additive disturbance to each of the four input-output equations, representing random errors in cost minimization, and specify that the four by one disturbance vector is independently and identically multi-

variate normally distributed with mean vector zero and nonsingular covariance matrix. As formulated above, the system of four equations is nonlinear in the parameters and variables. It is also simultaneous since $(\Delta K_t)^2 \equiv (K_{t+1} - K_t)^2$ appears as a right hand variable in (26), but ΔK_t is clearly endogenous in (29). Hence it would appear that we have a formidable estimation problem involving nonlinear simultaneity in both variables and parameters. On closer inspection, however, it is seen that the equation system is structurally recursive in that ΔK_t from (29) depends only on exogenous variables, and ΔK_t then enters only equation (26). As a result, the Jacobian matrix is unit triangular, and this potentially cumbersome term drops out of the simultaneous equations likelihood function, which therefore collapses to the likelihood function of the traditional nonlinear "seemingly unrelated" regression problem.[5]

The second application of the above model with a single quasi-fixed input involves disaggregating aggregate labor L into production, blue-collar, or unskilled labour (U) and nonproduction, white-collar or skilled labor (S). The resulting input-output demand equation system for this five input KUSEM model is strictly analogous to that for the KLEM model, as is the stochastic specification. We now turn to a discussion of data and empirical results for these two applications of the single quasi-fixed input model.

III. Empirical Results: One Quasi-fixed Input

In this section we briefly discuss data sources, present and interpret empirical results based on the KLEM and KUSEM models with one quasi-fixed input, and then outline a model with two quasi-fixed inputs.

A. Data

In the four input KLEM model with a single quasi-fixed factor (capital-denoted K), annual data on prices and quantities of the K, L, E, and M inputs are required, as are time series on net investment ($\dot{K}_t$), the real after tax rate of return (r_t), and the quantity of gross output (Q_t). While quarterly or monthly data would be preferable, at the present time only annual data are available for E and M. Our measure of output is production (including net change in inventories), not sales. In the five input KUSEM model the aggregate labor price and quantity data is decomposed into services of production or "unskilled" (U) and non-production, overhead, or "skilled" (S) labor.

Data sources, construction procedures and a listing of the annual 1947-76 time series on U, S, and L for annual total U.S. manufacturing are presented in the data appendix of E.R. Berndt and C.J. Morrison [1980]. Briefly, these labor data measure manhours at work (rather than hours paid for), and are adjusted for quality changes over time as measured by average educational attainment. The wage series includes employers' contributions to various supplementary benefit programmes. Aggregate labor price P_L is computed as a Divisia price index of P_U and P_S, and L is then obtained as total compensation to U and S divided by P_L.

Aggregate energy price and quantity data, 1947-71, are taken from Berndt-Fuss-Waverman [1980], as are data on M, P_M, K, P_K, $\dot{K}$, Q and r. Aggregate P_E is a Divisia price index of prices of crude petroleum, coal, refined petroleum products, electricity and natural gas, while P_M is a Divisia price index of prices of services from agriculture, construction, and mining; manufacturing excluding petroleum products; water and sanitary services, communications, trade and services; and transportation. E and M

are computed as total expenditures on E and M divided by P_E and P_M, respectively. P_K is a Divisia price index of capital rental prices for producer's durable equipment and nonresidential structures, taking account of variations in effective tax rates, and r is the real after-tax rate of return on all corporate capital in the U.S.[6] A detailed listing of all these variables, as well as further discussion of data sources and construction, is found in Chapter III of Berndt-Fuss-Waverman [1980].

B. Empirical Results: KLEM Model with K as Quasi-fixed Input

Using the total U.S. manufacturing data, 1952-71, parameters in the K, L, E, and M input-output equations were estimated using maximum likelihood techniques.[7] Parameter estimates are presented in Table 1, and elasticity estimates (for a representative year in the sample, 1961) are listed in Table 2. Elasticity estimates are calculated using the definitions in (13)-(15). The R^2 is good in the L equation, but is rather small in the M equation. Durbin-Watson statistics indicate that some autocorrelation may be present, although at the .01 level of significance, the test statistic in each equation falls in the "inconclusive" region.[8]

As seen in Table 2, our estimated dynamic KLEM model with a single quasi-fixed input generates elasticities consistent with SRIRL; the short run labor elasticity estimate with respect to output (ε^S_{LQ}) is 0.765 while the corresponding long run elasticity value (ε^L_{LQ}) is 1.000, implying increasing returns to aggregate labor in the short run but constant returns in the long run. This finding that $\varepsilon^S_{LQ} < \varepsilon^L_{LQ}$ requires, as was shown in the previous section, that L and K be long run complements. Our 1961 estimates of ε^I_{LK} and ε^L_{LK} are -0.014 and -0.064, respectively, while the ε^I_{KL} and ε^L_{KL} estimates

TABLE 1

Maximum Likelihood Parameter Estimates of KLEM Model with K Quasi-Fixed

U.S. Manufacturing, 1952-1971

(Ratio of Parameter Estimate to Asymptotic Standard Error in Parentheses)

Parameter	Estimate	Parameter	Estimate	Parameter	Estimate
α_0	0.671	α_{Kt}	-0.016	γ_{KK}	67.610
	(6.70)		(-1.17)		(2.78)
α_{0t}	-0.005	α_{Et}	-0.001	γ_{EK}	0.371
	(-2.41)		(-1.52)		(4.41)
α_K	-4.448	α_{Mt}	-0.006	γ_{MK}	-2.617
	(-2.21)		(-3.14)		(-7.03)
α_E	0.085	γ_{EE}	-0.026	$\ddot{\gamma}_{KK}$	773.188
	(2.37)		(-1.67)		(2.70)
α_M	1.255	γ_{MM}	-0.047	γ_{EM}	-0.003
	(12.08)		(-0.77)		(0.14)

Log of likelihood function: 211.591

	R^2	Durbin-Watson
L Equation	.9793	1.1292
E Equation	.7765	.8519
M Equation	.1908	.7381
$\dot{K}$ Equation	.6258	1.4866

TABLE 2

1961 Short, Intermediate and Long Run Elasticity Estimates
U.S. Manufacturing, KLEM Model with K Quasi-Fixed
Based on 1952-1971 Parameter Estimates

Elasticity	SR	IR	LR
ε_{KK}	.000	-.057	-.207
ε_{KL}	.000	-.074	-.265
ε_{KE}	.000	-.021	-.075
ε_{KM}	.000	.152	.547
ε_{LK}	.000	-.014	-.064
ε_{LL}	-.265	-.283	-.347
ε_{LE}	.093	.088	.070
ε_{LM}	.172	.210	.341
ε_{EK}	.000	-.029	-.110
ε_{EL}	.566	.528	.425
ε_{EE}	-.511	-.521	-.551
ε_{EM}	-.055	.022	.235
ε_{MK}	.000	.016	.059
ε_{ML}	.076	.096	.152
ε_{ME}	-.004	.002	.017
ε_{MM}	-.072	-.114	-.228
ε_{KQ}	.000	.277	1.000
ε_{LQ}	.765	.820	1.000
ε_{EQ}	.498	.639	1.000
ε_{MQ}	1.269	1.193	1.000

are -0.074 and -0.265. Table 2 also indicates that K and E are complements, with this complementarity increasing in the long run; L and E are substitutes, L and M are substitutes, and E and M are short run complements but long run substitutes. Although all own-price elasticities are larger (in absolute value) in the long run than in the short-run, the E-L cross elasticities indicate greater substitutability in the short than in the long run.

In terms of other output elasticities, overshooting occurs with M, $\varepsilon^S_{MQ} = 1.269$ and $\varepsilon^L_{MQ} = 1.000$, which is due to substantial K-M substitutability. On the other hand, the ε^S_{EQ} estimate of 0.498 is less than the long run elasticity value (ε^L_{EQ}) of 1.000, so that there are also substantial short run increasing returns to energy. This pattern of results suggests that in response to an increase in demand for its output, the dynamic cost-minimizing firm with K fixed in the short run responds first by increasing more than proportionally its purchases of intermediate and semi-processed inputs M, and then begins "producing" these intermediate goods internally by gradually increasing its use of K, L and E inputs, but decreasing purchases of M. Note that this result on M is net of inventory changes, for Q is production, not sales. It is also of interest to note that the estimated annual rate of total factor productivity [the negative of $(\partial G/\partial t)/G$] is 0.86% in 1961. Finally, the 1961 estimate of m^*_{11} -- the proportional endogenous adjustment coefficient for capital -- is 0.266, indicating that due to increasing marginal costs of adjustment, the dynamic cost-minimizing firm adjusts its capital stock slowly.

C. Empirical Results: KUSEM Model with K as Quasi-Fixed Input

Although the above KLEM results resulted in estimated SRIRL, they required aggregate labor-capital complementarity, which is unappealing *a priori*. Since there is some evidence[9] suggesting that non-production, overhead or "skilled" labor (S) is complementary with K while production or "unskilled" labor (U) is substitutable with K, it may be reasonable to expect that there are substantial short run increasing returns to skilled labor and short run constant or decreasing returns to unskilled labor.[10] Moreover, the small value of ε^{S}_{SQ} may result in an aggregate labor-output elasticity consistent with SRIRL.

In order to examine this aggregation hypothesis further, we have decomposed aggregate labor L into S and U, specified input-output demand equations for the variable inputs S, E, and M based on (25), an input-output demand equation for the normalized U factor based on (26), and an input-output demand equation for the quasi-fixed capital input as in (29). Maximum likelihood parameter estimates of this KUSEM model are presented in Table 3, while 1961 elasticity estimates appear in Table 4. R^2 figures are improved when compared to the KLEM model, and Durbin-Watson statistics remain in the inconclusive region at the .01 level of significance.

A striking result here is that the output elasticity for production or unskilled labor is above unity in the short run (1.376), while the short run output elasticity for overhead, nonproduction or skilled labor is much smaller, 0.165; both, however, become unity in the long run. When these two types of labor are aggregated, the implied elasticity of demand for aggregate (S+U) labor with respect to output is 0.898 in the short run and 1.000 in the long run (see bottom of Table 4). Thus the increasing returns to aggregate

TABLE 3

Maximum Likelihood Parameter Estimates of KUSEM Model with K Quasi-Fixed
U.S. Manufacturing, 1952-1971
(Ratio of Parameter Estimate to Asymptotic Standard Error in Parentheses)

Parameter	Estimate	Parameter	Estimate	Parameter	Estimate
α_0	0.306 (2.62)	α_{St}	-0.003 (-1.17)	γ_{EM}	-0.039 (-2.25)
α_{0t}	-0.001 (-0.30)	α_{Et}	-0.001 (-1.95)	γ_{MM}	-0.082 (-1.14)
α_K	-6.318 (-3.64)	α_{Mt}	-0.008 (-3.75)	γ_{KK}	56.283 (2.64)
α_S	0.425 (2.81)	γ_{SS}	-0.267 (-1.90)	γ_{SK}	1.882 (4.62)
α_E	0.130 (4.00)	γ_{SE}	-0.048 (-1.78)	γ_{EK}	0.630 (7.79)
α_M	1.369 (15.16)	γ_{SM}	-0.024 (-0.29)	γ_{MK}	-2.230 (-5.34)
α_{Kt}	-0.001 (-0.09)	γ_{EE}	-0.004 (-0.33)	$\gamma_{\ddot{K}\ddot{K}}$	704.321 (2.71)

Log of likelihood function: 262.296

	R^2	Durbin-Watson
U Equation	.9615	1.3710
S Equation	.7933	.7649
E Equation	.8503	1.3913
M Equation	.2289	.7598
$\dot{K}$ Equation	.5848	1.4892

TABLE 4

1961 Short, Intermediate and Long Run Elasticity Estimates
U.S. Manufacturing, KUSEM Model with K Quasi-Fixed
Based on 1952-1971 Parameter Estimates

Elasticity	SR	IR	LR	Elasticity	SR	IR	LR
ε_{KK}	0.000	-.064	-.242	ε_{EK}	0.000	-.054	-.215
ε_{KU}	0.000	.048	.183	ε_{EU}	1.519	1.560	1.681
ε_{KS}	0.000	-.090	-.338	ε_{ES}	-.686	-.762	-.985
ε_{KE}	0.000	-.040	-.149	ε_{EE}	-.071	-.104	-.203
ε_{KM}	0.000	.145	.546	ε_{EM}	-.762	-.640	-.278
ε_{UK}	0.000	.023	.073	ε_{MK}	0.000	.015	.059
ε_{UU}	-2.282	-2.300	-2.338	ε_{MU}	.207	.196	.163
ε_{US}	1.090	1.123	1.192	ε_{MS}	-.027	-.006	.056
ε_{UE}	.422	.437	.468	ε_{ME}	-.057	-.048	-.021
ε_{UM}	.770	.717	.604	ε_{MM}	-.123	-.157	-.257
ε_{SK}	0.000	-.055	-.216	ε_{KQ}	0.000	.266	1.000
ε_{SU}	1.739	1.780	1.901	ε_{UQ}	1.376	1.257	1.000
ε_{SS}	-1.276	-1.352	-1.577	ε_{SQ}	.165	.390	1.000
ε_{SE}	-.304	-.338	-.437	ε_{EQ}	.169	.394	1.000
ε_{SM}	-.158	-.035	.329	ε_{MQ}	1.228	1.167	1.000

Implied Elasticity of Total Labor With Respect to Output in 1961:	SR	IR	LR
	.898	.915	1.000

labor in the short run appear to be the result of strongly increasing returns to nonproduction manhours, which overrides the decreasing returns to production manhours. The estimated cross-price elasticities between S and K and between U and K also corroborate our *a priori* hypotheses: (i) S and K are complements -- $\varepsilon^{I}_{SK} = -0.055$, $\varepsilon^{L}_{SK} = -0.216$, $\varepsilon^{I}_{KS} = -0.090$, and $\varepsilon^{L}_{KS} = -0.338$, while (ii) U and K are substitutes -- $\varepsilon^{I}_{UK} = 0.023$, $\varepsilon^{L}_{UK} = 0.073$, $\varepsilon^{I}_{KU} = 0.048$ and $\varepsilon^{L}_{KU} = 0.183$.

In terms of the other output elasticities, while $\varepsilon^{S}_{KQ} = 0$ in the short run (due to quasi-fixity of K), in the intermediate run this elasticity equals 0.266, and 1.000 in the long run. A somewhat surprising result is the rather small energy elasticity -- $\varepsilon^{S}_{EQ} = 0.169$ and $\varepsilon^{I}_{EQ} = 0.394$. This implies that in the short run as output is increased and K is fixed, E demand increases only slightly -- perhaps due to substantial space and process heating requirements being independent of short run output trends. On the other hand, the output elasticity estimate for M again indicates short run overshooting, for $\varepsilon^{S}_{MQ} = 1.228$.

In summary, output elasticity estimates from this KUSEM model suggest that in the short run when K is fixed and in response to an exogenous increase in output, demand for intermediate and partially finished goods M increases more than proportionally -- since M is highly substitutable with K --as does demand for production worker manhours. On the other hand, demands for overhead labor (S) and for energy (E) are similar and quite unresponsive as a result of E-K and S-K complementarity, indicating substantial short run increasing returns to E and S. In the longer term as K adjusts to its steady state level, the cost-minimizing firm gradually reduces its M intensity and "produces" some of these intermediate inputs and semi-finished goods internally with additional inputs of K, U, and E.

Although a host of elasticity estimates are presented in Table 4, a few are worthy of particular attention. Similar to the KLEM model, we find that E and K are complements, and that this complementarity increases in the long run. Energy price increases affect the two labor types differently -- increasing the demand for production workers but decreasing the derived demand for overhead labor S. The short run price elasticity ε^S_{UE} of 0.422 indicates that average productivity of production workers will fall as a result of energy price increases, while the ε^S_{SE} estimate of -0.304 indicates increasing average productivity to nonproduction labor. The effect of energy price increases on aggregate (S + U) labor therefore depends on which effect dominates; since the level of U manhours is greater than S manhours, and since the ε^S_{UE} elasticity is larger in absolute value than the ε^S_{SE} elasticity, it follows that increasing energy prices depress the average productivity of aggregate S + U labor. It is also worth noting that all own price elasticity estimates are larger (in absolute value) in the long than in the short run, and that these estimates are quite similar in the KLEM and KUSEM models.[11] The estimated annual rate of total factor productivity in 1961 is 0.86%, exactly the same as in the KLEM model.

Although the above empirical results for the KUSEM model are *a priori* plausible and reasonably satisfactory and in particular again yield SRIRL as the outcome of a cost minimization process, the underlying model still fails to account fully for the quasi-fixity of non-production or overhead labor. We now briefly outline an extension of our KUSEM model with a single quasi-fixed input -- K -- to a model with two quasi-fixed inputs -- S and K. Empirical results for this extended model are then presented in Section V.

IV. Formulation of KUSEM Model with Two Quasi-Fixed Inputs

As noted above, the Treadway internal cost of adjustment model specifies that demand for the quasi-fixed factors can be characterized by the system of differential equations (9) and (10). In the case of two quasi-fixed factors, x_1 and x_2, the system of two differential equations can be written as

$$\dot{x}_1 = m^*_{11}(x^*_1 - x_1) + m^*_{12}(x^*_2 - x_2)$$

$$\dot{x}_2 = m^*_{21}(x^*_1 - x_1) + m^*_{22}(x^*_2 - x_2) \quad . \qquad \ldots . (30)$$

The solution values for M* are, however, extremely complex, and depend on r and the derivatives of the cost function G. In order to make our problem analytically and empirically tractable, we make the simplifying assumption that M* is a diagonal matrix, i.e., that $m^*_{12} = m^*_{21} = 0$. Let $x_1 = K$ and $x_2 = S$. Necessary and sufficient conditions for M* to be diagonal are that $G_{x_1x_2} = G_{x_2x_1} = G_{\dot{x}_1\dot{x}_2} = G_{\dot{x}_2\dot{x}_1} = 0$, which from (24) implies that $\ddot{\gamma}_{12} = \ddot{\gamma}_{21} = 0$.[12] These conditions can also be interpreted as requiring that long run price elasticities between S and K are equal to zero. Under such independence, the adjustments of S and K are independent of one another, for the marginal product of S(K) is not affected by changes in the level of K(S).[13] When $m^*_{12} = m^*_{21} = 0$ is imposed on the system (10), the solution values for m^*_{ii} becomes

$$m^*_{ii} = -\frac{1}{2}\left[r - \left(r^2 + 4(G_{x_ix_i}/G_{\dot{x}_i\dot{x}_i})\right)^{1/2}\right] \quad , \qquad i = 1, 2 \ldots . (31)$$

In order to obtain estimating equations, it is again necessary to adopt a specific functional form. Using the normalized variable cost function

(24) in the KUSEM model with S and K as quasi-fixed inputs, and imposing the restriction $\ddot{\gamma}_{SK} = \gamma_{SK} = 0$ we use (8) and obtain estimating equations for the S and K quasi-fixed inputs analogous to (29). The short run input-output demand equations for the variable inputs E and M are obtained using (25), while the normalized input-output demand equation for U is based on (26).

Elasticity formulations differ when the number of quasi-fixed inputs is increased from one to two. Among variable inputs, short run price elasticities are

$$\varepsilon^S_{v_j w_k} = (w_k/v_j) \cdot \left(\left. \frac{\partial v_j}{\partial w_k} \right|_{K=\bar{K},\ S=\bar{S}} \right), \quad j,k = U,E,M \qquad \ldots .(32)$$

while the corresponding long run price elasticities are computed as

$$\varepsilon^L_{v_j w_k} = (w_k/v_j) \cdot \left(\left. \frac{\partial v_j}{\partial w_k} \right|_{K=\bar{K},\ S=\bar{S}} + \frac{\partial v_j}{\partial K^*} \frac{\partial K^*}{\partial w_k} + \frac{\partial v_j}{\partial S^*} \cdot \frac{\partial S^*}{\partial w_k} \right) \quad \ldots .(33)$$

Short run price elasticities between fixed and variable inputs are zero by assumption, while long run elasticities are defined as

$$\varepsilon^L_{x_i w_k} = (w_k/x_i) \cdot \left(\frac{\partial x_i^*}{\partial w_k} \right), \qquad i = S,K; \quad k = U,E,M \qquad \ldots .(34)$$

Output elasticities for variable inputs can be represented as

$$\varepsilon^S_{v_j Q} = \left(\frac{Q}{v_j} \right) \cdot \left(\left. \frac{\partial v_j}{\partial Q} \right|_{K=\bar{K},\ S=\bar{S}} \right) \qquad \ldots .(35)$$

in the short run and

$$\varepsilon^{L}_{v_jQ} = (Q/v_j) \cdot \left(\frac{\partial v_j}{\partial Q}\bigg|_{K=\bar{K},S=\bar{S}} + \frac{\partial v_j}{\partial K^*} \cdot \frac{\partial K^*}{\partial Q} + \frac{\partial v_j}{\partial S^*} \cdot \frac{\partial S^*}{\partial Q} \right) ,$$

$$j = U,E,M \qquad \ldots .(36)$$

in the long run. For the two quasi-fixed inputs, short run demand elasticities with respect to output are zero, while long run elasticities are unity by assumption. Finally, long run cross-price elasticities among quasi-fixed inputs are

$$\varepsilon^{L}_{x_iu_j} = (u_j/x_i^*)(\partial x_i^*/\partial u_j) , \qquad i,j = K,S \ . \qquad \ldots .(37)$$

For all the above elasticities, intermediate (one time period) elasticities are defined by accounting for the partial adjustment m^*_{SS} of S to S* and m^*_{KK} of K to K*.

Before turning to a discussion of empirical results, we briefly re-examine the SRIRL phenomenon in the context of our two quasi-fixed input model. Since ε^{S}_{SQ} (the output elasticity of demand for skilled labor in the short run) is zero by assumption, and since $\partial S^*/\partial Q$ is positive in the long run, it follows that $\varepsilon^{L}_{SQ} - \varepsilon^{S}_{SQ}$ is positive. Labor hoarding, as incorporated in the adjustment cost model, ensures that this inequality holds. However, for unskilled labor which is assumed here to be a variable input, the difference between long and short run output elasticities, using (35) and (36) turns out to be

$$\varepsilon^{L}_{UQ} - \varepsilon^{S}_{UQ} = (Q/U) \cdot \left(\frac{\partial U}{\partial K^*} \cdot \frac{\partial K^*}{\partial Q} + \frac{\partial U}{\partial S^*} \cdot \frac{\partial S^*}{\partial Q} \right) . \qquad \ldots .(38)$$

Regularity conditions on the production (cost) function require that $\partial K^*/\partial Q$ and $\partial S^*/\partial Q$ be strictly positive. However, since these regularity conditions

impose no restrictions on the sign of $\partial U/\partial K^*$ and $\partial U/\partial S^*$, the difference $\varepsilon^L_{UQ} - \varepsilon^S_{UQ}$ can be negative, zero, or positive. Sufficient conditions for $\varepsilon^L_{UQ} - \varepsilon^S_{UQ}$ to be positive (negative) are that U and K, as well as U and S are complementary (substitutable) inputs.

Let us define the sum $S + U$ as aggregate labor $\tilde{L}$. The difference between long and short run elasticities of demand for aggregate labor $\tilde{L}$ can be written as

$$\varepsilon^L_{\tilde{L}Q} - \varepsilon^S_{\tilde{L}Q} = \left(\frac{Q}{\tilde{L}}\right)\cdot\left[\left(\frac{\partial U}{\partial Q} + \frac{\partial S}{\partial Q}\right)^L - \left(\frac{\partial U}{\partial Q} + \frac{\partial S}{\partial Q}\right)^S\right]$$

$$= (Q/\tilde{L})\cdot\left(\frac{\partial U}{\partial K^*}\cdot\frac{\partial K^*}{\partial Q} + \frac{\partial U}{\partial S^*}\cdot\frac{\partial S^*}{\partial Q} + \frac{\partial S^*}{\partial Q}\right) . \qquad \ldots .(39)$$

Hence, again, a sufficient condition for $\varepsilon^L_{\tilde{L}Q} - \varepsilon^S_{\tilde{L}Q}$ to be positive is that there be U-K and U-S complementarity ($\partial S^*/\partial Q$ is positive by assumption). On the other hand, even if unskilled labor were substitutable with both physical capital and with skilled labor, as long as $\partial S^*/\partial Q$ were sufficiently large, the short run aggregate labor-output elasticity would still be less than the corresponding long run elasticity. This implies that SRIRL in the aggregate is possible even with U-K and U-S substitutability. It is clear, therefore, that the mere existence of labor hoarding (or quasi-fixity) of skilled labor is not sufficient to yield the classic result that $\varepsilon^S_{\tilde{L}Q} < \varepsilon^L_{\tilde{L}Q}$. If for example, U and K, and U and S were highly substitutable and $\partial S^*/\partial Q$ were relatively small, even with labor hoarding of S it would be possible for $\varepsilon^S_{\tilde{L}Q} > \varepsilon^L_{\tilde{L}Q}$.

V. Empirical Results: KUSEM Model with K and S as Quasi-Fixed Inputs

We now turn to a discussion of empirical results obtained with the KUSEM model for U.S. Manufacturing, 1952-71, based on two quasi-fixed inputs, K and S. Parameters were again estimated using the method of maximum likelihood. These estimates are presented in Table 5 below, while implied elasticity estimates for 1961 are listed in Table 6. Note that the log-likelihood function at the sample maximum is considerably larger with the two quasi-fixed input specification than with the single quasi-fixed factor KUSEM model. The R^2 figures in this model are improved in the U, E and M equations from the only-K-fixed KUSEM model, although the $\dot{K}$ fit is decreased. Durbin-Watson statistics again are in the inconclusive region at the .01 level of significance.

As seen in Table 5, parameter estimates of γ_{SS} and γ_{KK} are each statistically significant, implying that adjustment costs are significant for both S and K, thereby lending support to the two quasi-fixed input specification. A rather interesting additional implication of the parameter estimates is that the cost-minimizing adjustment coefficients m^*_{SS} amd m^*_{KK} are both small; 1961 estimated values are 0.100 and 0.220, respectively, suggesting that both inputs adjust slowly to their long run equilibrium levels.[14]

In Table 6,we see that output elasticities vary considerably among the inputs. Again there is "overshooting" in the short run with M; the short run elasticity of demand for unskilled labor with respect to output (1.349) is also greater than the long run unity value, while for energy there are substantial increasing returns in the short run. For skilled labor, the intermediate run elasticity estimate is quite small (0.123), as is that for capital (0.235).

TABLE 5

Maximum Likelihood Parameter Estimates of KUSEM Model

S and K Quasi-Fixed, U.S. Manufacturing, 1952-1971

(Ratio of Parameter Estimate to Asymptotic Standard Error in Parentheses)

Parameter	Estimate	Parameter	Estimate	Parameter	Estimate
α_0	0.477 (6.13)	α_{St}	-0.010 (-2.89)	γ_{SS}	0.698 (1.58)
α_{0t}	-0.003 (-1.46)	α_{ET}	-0.001 (-1.81)	$\ddot{\gamma}_{KK}$	458.550 (2.99)
α_K	-3.020 (-4.17)	α_{Mt}	-0.007 (-4.96)	$\ddot{\gamma}_{SS}$	43.568 (1.99)
α_S	-0.374 (-1.82)	γ_{EE}	-0.010 (-0.74)	γ_{EK}	0.326 (3.70)
α_E	0.080 (2.71)	γ_{EM}	-0.029 (-2.01)	γ_{ES}	0.113 (3.07)
α_M	1.326 (17.14)	γ_{MM}	-0.068 (-1.52)	γ_{MK}	-1.179 (-3.76)
α_{Kt}	.009 (0.78)	γ_{KK}	28.495 (3.87)	γ_{MS}	-0.574 (-4.59)

Log of Likelihood function: 286.101

	R^2	Durbin-Watson
U Equation	.9748	1.2723
E Equation	.8815	1.5174
M Equation	.2820	0.7626
$\dot{S}$ Equation	.2873	2.1706
$\dot{K}$ Equation	.4279	1.2727

TABLE 6

1961 Short, Intermediate, and Long-Run Elasticity Estimates
U.S. Manufacturing, KUSEM Model with K, S Quasi-Fixed
Based on 1952-1971 Parameter Estimates

Elasticity	SR	IR	LR	Elasticity	SR	IR	LR
ε_{KK}	0.000	-.112	-.511	ε_{EK}	0.000	-.048	-.219
ε_{KU}	0.000	.015	.067	ε_{EU}	.740	.829	1.608
ε_{KS}	0.000	0.000	0.000	ε_{ES}	0.000	-.228	-2.296
ε_{KE}	0.000	-.036	-.162	ε_{EE}	-.183	-.233	-.596
ε_{KM}	0.000	.133	.606	ε_{EM}	-.556	-.320	1.503
ε_{UK}	0.000	.010	.025	ε_{MK}	0.000	.014	.062
ε_{UU}	-.734	-.796	-1.304	ε_{MU}	.146	.111	-.197
ε_{US}	0.000	.167	1.550	ε_{MS}	0.000	.091	.914
ε_{UE}	.204	.232	.444	ε_{ME}	-.042	-.024	.114
ε_{UM}	.530	.387	-.715	ε_{MM}	-.104	-.191	-.894
ε_{SK}	0.000	0.000	0.000	ε_{KQ}	0.000	.235	1.000
ε_{SU}	0.000	.241	2.425	ε_{UQ}	1.349	1.287	1.000
ε_{SS}	0.000	-.658	-6.632	ε_{SQ}	0.000	.123	1.000
ε_{SE}	0.000	-.098	-.991	ε_{EQ}	.226	.369	1.000
ε_{SM}	0.000	.516	5.198	ε_{MQ}	1.259	1.214	1.000

Implied elasticity of Total Labor With Respect to Output in 1961:

SR	IR	LR
.803	.816	1.000

As noted in the previous section, the mere presence of labor hoarding of S is not sufficient to yield the result that for aggregate labor $\tilde{L} = U + S$, $\varepsilon^{L}_{\tilde{L}Q} > \varepsilon^{S}_{\tilde{L}Q}$. Based on our parameter estimates, we find that the elasticity of aggregate labor with respect to output is less than unity ($\varepsilon^{S}_{\tilde{L}Q} = 0.803$), implying again that in the short run there are increasing returns to aggregate labor, although by assumption the corresponding long run elasticity is unity. Hence, our two quasi-fixed input model provides empirical results completely consistent with SRIRL.

A number of other results deserve special mention. Again we find that the own price elasticities are larger (in absolute value) in the long run than in the short run. Moreover, the difference between short and long run own price elasticities is larger than was the case with the KLEM and KUSEM simple one quasi-fixed input specifications. Short run substitution between U and M is greater, however, than in the long run. The two types of labor are quite substitutable in the long run ($\varepsilon^{L}_{SU} = 2.425$, $\varepsilon^{L}_{US} = 1.550$), and E and K indicate increasing complementarity as K and S adjust to their long run equilibrium levels. Inputs E and M are short run complements, but long run substitutes. Finally, the estimated annual rate of total factor productivity in 1961 is 0.98%.

In summary, we find that when K and S are both treated as quasi-fixed inputs, we again obtain short run increasing returns to aggregate labor as the outcome of a dynamic cost-minimizing process. Moreover, the difference between short and long run own-price elasticity estimates increases compared to estimates based on K as the only quasi-fixed input.

VI. Concluding Remarks

An underlying theme of this paper is that SRIRL can be viewed as the natural outcome of a dynamic cost minimization process. Although this view has traditionally been associated with the notion of labor hoarding, we have demonstrated analytically and empirically that labor hoarding is neither necessary nor sufficient for SRIRL. In these concluding remarks we relate our approach and results to other relevant literature, and then draw attention to important issues meriting further research.

The analytical model used in this paper is based on explicit dynamic cost minimization with myopic expectations. This framework differs from the well-known "disequilibrium interrelated factor demand" model of M.I. Nadiri and Sherwin Rosen [1969, 1973] in several respects: (i) the partial adjustment matrix M* is an exogenous constant parameter matrix in the Nadiri-Rosen approach, but is endogenous and variable in our dynamic cost minimization framework. (ii) In Nadiri-Rosen, lagged values of all factor demands appear as regressors in each of the input demand equations, a specification which they interpret as an "approximation" [1969, p. 459] to the underlying differential equation solution. By contrast, the explicitly derived closed form input demand equations of this paper have lagged values of only the quasi-fixed factors as regressors in each of the factor demand equations, and lagged values of variable inputs do not appear as regressors anywhere in the interrelated demand equation system. Hence, in our framework the M* matrix has smaller dimension and requires estimation of fewer parameters. (iii) On the other hand, implementation of our model requires distinguishing variable from quasi-fixed inputs; such a distinction is not necessary in the Nadiri-Rosen approach.

Although our interrelated demand approach is rich in economic structure, it is based on the restrictive assumption of myopic expectations for output and relative factor prices. One possible alternative approach, used by Alan D. Woodland [1977], is to specify that input quantity adjustment is instantaneous but that price expectations for each input are a deterministic distributed lag of previous input prices. A much more ambitious approach would be to specify dynamic cost minimization with stochastic expectations for output and all input prices based on some form of rationality; such an approach would have the additional advantage of differentiating temporary from permanent changes in output or factor prices, a distinction not possible in our myopic expectations framework. The virtue of our approach is that it is still rich in economic structure and is empirically implementable; as noted in the introduction, recent research by Hansen and Sargent [1979, 1980] suggests that convenient and empirically implementable mathematical representations for multiple input dynamic factor demand models with rational expectations are not yet available and require further development.[15] It is worth noting, however, that we expect our analytic result of labor hoarding being neither necessary nor sufficient for SRIRL will carry over to dynamic models with greater stochastic content.

Future research on dynamic factor demand models might focus on a number of other issues as well: (i) we have assumed that output is exogenous, an assumption consistent with the Granger causality results reported by Sims [1974] and Sargent [1978]. Although this assumption could be relaxed, we would expect some difficulties in modelling firm and market output pricing in the presence of internal costs of adjustment; for example strict short run marginal cost pricing would not seem plausible.[16]

(ii) In order to obtain a closed form empirically implementable solution, we have assumed that the partial adjustment matrix M* is diagonal. Some recent work by Larry Epstein [1979] and Epstein-Michael Denny [1980] suggests that if one is willing to make the assumption that M* is constant, then the diagonality assumption can be relaxed. Preliminary econometric results reported by Denny-Epstein with such a model indicate that considerable additional research may still be necessary. (iii) The dynamic model might be amended so that gross rather than net changes in the quasi-fixed factors affect costs of adjustment. One attractive feature of such a model, noted by F.P.R. Brechling [1975], would be that firm size is determinate, even in the long run with constant returns to scale. (iv) In terms of econometric implementation, we have jumped from a continuous time theoretical model to a discrete time implementation, and much remains to be done in assessing robustness of results to alternative discrete time representations, to more complicated stochastic specifications, and to varying choice of the normalized input.[17] (v) Finally, although our annual time series data for total U.S. manufacturing have provided us with a reasonably plausible illustration of factors affecting SRIRL in a dynamic model, it would be preferable to use monthly, quarterly or even pooled cross-section, time series data to obtain more reliable estimates. Unfortunately, at the present time only annual data are available on a consistent basis for Q and the K, L, E and M inputs in U.S. manufacturing.

FOOTNOTES

[1] For a review of this literature, see Ray C. Fair [1969] and C.J. Morrison and E.R. Berndt [1979].

[2] In a later article, however, Solow [1973] goes one step further and conjectures that utilized capital may be a more variable input than aggregate labor in the short run, and that this yields observed SRIRL. In his empirical analysis, Solow divides output by the Wharton measure of capacity utilization and multiplies measured capital stock by the Christensen-Jorgenson [1969] index of capital utilization. Solow then inserts this cyclically adjusted data into a static Cobb-Douglas production function, and obtains results consistent with diminishing marginal products and constant returns to scale.

[3] For an intuitive discussion of internal costs of adjustment, see F.P.R. Brechling and Dale T. Mortenson [1971].

[4] Walter Oi [1962, p. 539] already recognized this, for he stated that ". . . the variable factors that are most substitutable with the fixed factors will exhibit the greatest relative shifts in factor demands."

[5] For further discussion, see Catherine J. Morrison [1979].

[6] Prices are normalized by P_L in the KLEM formulation and P_U in the KUSEM specification.

[7] Computations were carried out using the Time Series Processor on the Amdahl 470 V/6-II computer at the University of British Columbia.

[8] Note, however, that the $\dot{K}$ equation contains a lagged value of the capital stock, and that $\dot{K}$ appears as a regressor in the L equation, thereby, reducing the reliability of the Durbin-Watson test statistic.

[9] See, for example, the classic article by Zvi Griliches [1969] and the survey by Daniel Hamermesh and James Grant [1979].

[10]Recall that short run (within one year) constant returns to production labor have been reported by Christopher A. Sims [1974].

[11]The 1961 estimate of m_{11}^{*} in this model is .253 (with a standard error estimate of .054), almost the same as in the KLEM model, indicating that the firm adjusts its capital stock slowly.

[12]See Dale T. Mortenson [1973, p. 660] and Larry G. Epstein [1980].

[13]Although based on different assumptions and model formulations, the empirical evidence on S-K substitutability has provided both positive and negative S-K elasticity estimates; see, for example, the survey by Hamermesh and Grant [1979].

[14]Standard error estimates for m_{SS}^{*} and m_{KK}^{*} are .065 and .050, respectively, implying that their 95% confidence intervals overlap.

[15]See S.J. Nickell [1977] for further discussion. A particularly interesting specification is that advocated by J. Helliwell and G. Glorieux [1970] with input relative price expectations myopic, but output expectations adaptive and stochastic.

[16]See Robert M. Solow [1968].

[17]Recall from Tables 1, 3 and 5 that there was some evidence indicating possible autocorrelation in some of the equations.

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